FLOW OF AN IDEAL FLUID WITH FREE SURFACES IN DOUBLY CONNECTED AND TRIPLY CONNECTED REGIONS

(TECHENIIA IDEAL'NOI ZHIDKOSTI SO SVOBODNYMI POVERKHNOSTIAMI V DVUKHSVIAZNYKH I TREKHSVIAZNYKH OBLASTIAKH)

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Below we investigate a method of solution of the plane problem in the dynamics of an ideal, incompressible fluid without gravity forces, of steady, separated flow over bodies of polygonal form for the case where the flow field is doubly or triply connected. It is shown that the solution of the given problem reduces to quadratures. The solution of the analogous problem for a singly connected region may be obtained by the method of Joukowski [1-3], and for a doubly connected region by the method of Sedov [2]. By way of illustration of the method, the solution for a submerged wing is obtained.

1. Formulation of the problem. Let the steady, plane potential flow of an ideal incompressible fluid without gravity forces occupy a triply connected (or doubly connected) region in the physical plane of the complex variable $z = x + iy$. Part of the boundary of the flow is known *a priori*, and is composed of straight-line segments; we denote it by M. The remaining part of the boundary of the flow is free and is unknown to start with; we denote it by L.

In the plane hydrodynamics of steady, potential flows of an incompressible ideal fluid, the following basic relations hold [1,3]:

$$v_x + iv_y = \overline{w'(z)}, \qquad p = A - {}^1/_2\rho\,|w'(z)|^2 \qquad (A = p_\infty + {}^1/_2\rho v_\infty{}^2) \quad (1.1)$$

Here, p is the pressure, v_x and v_y are the velocity components in the directions of the coordinate axes, ρ is the density, p_∞ and v_∞ are the pressure and velocity at infinity.

The analytic function $w(z)$ may be represented in the form

$$w(z) = \sum_{k=1}^{3} \frac{\Gamma_k}{2\pi i} \ln (z - z_k) + w_0(z) \tag{1.2}$$

Here, Γ_k is the circulation around the boundary contour C_k, $k = 1$, 2, 3; z_k is a point inside the contour C_k; the function $w_0(z)$ is single-valued, analytic; the quantities Γ_k are given or are determined from certain additional conditions.

With the help of the basic relations (1.1), the boundary conditions of the problem may be formulated in the following form:

$$\operatorname{Im} w(z) = \psi_k \quad \text{on } C_k \qquad (k = 1, 2, 3) \tag{1.3}$$

$$\arg w'(z) = -\theta_j \quad \text{on } M, \qquad |w'(z)|^2 = v_\infty^2 (1 + Q_i) \quad \text{on } L \tag{1.4}$$

Here, θ_j is the angle made by the jth straight-line segment of the boundary M with the flow direction along the x-axis; Q_i are the cavitation numbers on the various portions of the free boundary. One of the constant and *a priori* unknown quantities ψ_k may be put equal to zero.

Similar mathematical problems arise in the theory of cavitation and jets, in the theory of a submerged wing, etc. [1-3]. In fact, when the segment L is absent, we obtain the problem of potential flow over three polygons; when the contour C_k is part of the boundary L, it may be interpreted as a hollow vortex, etc.

2. Solution of boundary value problems. 1. We go into the parametric plane of the complex variable ζ by means of the transformation $z = \omega(\zeta)$. The function $z = \omega(\zeta)$ gives a single-sheeted conformal mapping of the exterior of three (or two) cuts on the real axis of the ζ-plane onto the triply (or doubly) connected region of flow in the z-plane, with the point at infinity of the z-plane corresponding to a certain point $\zeta = \zeta_0$ of the ζ-plane. It is well known that such a mapping is always possible for arbitrary doubly connected and triply connected regions [4,5]. We note that, generally speaking, a region of connectivity of order higher than three cannot be mapped on the exterior of cuts along a straight line [4,5].

Define

$$w[\omega(\zeta)] = \Psi(\zeta), \qquad \ln \frac{w'[\omega(\zeta)]}{v_\infty} = \Phi(\zeta) \tag{2.1}$$

The boundary value problems for determining the analytic functions $\Phi(\zeta)$ and $\Psi(\zeta)$ on the basis of equations (1.3) and (1.4) may be written in the form

$$\operatorname{Im} \Psi(\zeta) = \psi_k \quad \text{for } \zeta \in (a_k, b_k) \qquad (k = 1, 2, 3) \tag{2.2}$$

$$\operatorname{Re} \Phi(\zeta) = {}^1\!/_2 \ln(1 + Q_i) \quad \text{for } \zeta \in L, \qquad \operatorname{Im} \Phi(\zeta) = -\theta_j \quad \text{for } \zeta \in M \tag{2.3}$$

Here, L and M again denote the maps of the corresponding portions of the boundary in the z-plane.

The functions $\Phi(\zeta)$ and $\Psi(\zeta)$ possess the following properties.

1) If the point at infinity of the z-plane is an interior point and the expansion

$$w(z) = v_\infty e^{-i\theta} z + \frac{\Gamma}{2\pi i} \ln z + O(z^{-1})$$

is valid there, then at the corresponding point $\zeta = \zeta_0$ the function $\Psi(\zeta)$ will have a pole and a logarithmic singularity, while the function $\Phi(\zeta)$ will be bounded. If the point at infinity of the z-plane is a boundary point, then at the corresponding point of the ζ-plane there may be a logarithmic singularity or pole of the function $\Psi(\zeta)$. For example, the reentrant jet in the method of Efros and Gilbarg-Rock corresponds to a logarithmic singularity (and its neighborhood) in the ζ-plane.

2) Vortices and sources in the z-plane correspond to logarithmic singularities in the function $\Psi(\zeta)$ and poles in the function $\Phi(\zeta)$.

3) In making a traverse of the segment (a_k, b_k), corresponding to the contour C_k, the real part of the function $\Psi(\zeta)$ undergoes an increase Γ_k. At points of the ζ-plane which correspond to critical points of the flow, the function $\Phi(\zeta)$ has a logarithmic singularity.

4) At the ends of the segments (a_k, b_k) the functions $\Phi(\zeta)$ and $\Psi(\zeta)$ are, generally speaking, bounded.

A more detailed investigation of the types of singular points of the functions $\Phi(\zeta)$ and $\Psi(\zeta)$ which may be encountered is found in the monograph of Gurevich [3].

It is convenient to differentiate condition (2.2) with respect to ζ and to look for solutions of the resulting Dirichlet problem, in terms of single-valued functions which are now not bounded at the end points a_k, b_k. The solution of this problem is in [2, 6, 7].

The mixed boundary value problem (2.3) is a particular case of the Riemann-Hilbert boundary value problem with discontinuous coefficients for the exterior of cuts along a straight line, whose closed solution has been obtained by the author [8]. We note that in the case of a doubly connected region the original flow problem may be reduced to a

Dirichlet problem and a mixed boundary value problem for the circle, whose solution in closed form was obtained by Sedov [2].

2. The mixed boundary value problem (2.3) may be reduced [8] to a Riemann boundary value problem for two functions with a matrix coefficient $G(t)$

$$G(t) = \begin{vmatrix} 0 & \alpha(t) \\ \beta(t) & 0 \end{vmatrix}, \qquad \alpha(t) = \begin{cases} -1 & (t \in L^+) \\ +1 & (t \in M^+) \end{cases}, \qquad \beta(t) = \begin{cases} -1 & (t \in L^-) \\ +1 & (t \in M^-) \end{cases} \tag{2.4}$$

Here, $L^{\pm}$, $M^{\pm}$ denote the portions of the cuts lying along the upper or lower edges. The right-hand part of Riemann's boundary value problem can be written in the form

$$f = \begin{vmatrix} f_1(t) \\ f_2(t) \end{vmatrix}, \quad f_1(t) = \begin{cases} 2\ln(1 + Q_i) & (t \in L^+) \\ -2i\theta_j & (t \in M^+) \end{cases} \quad f_2(t) = \begin{cases} 2\ln(1 + Q_i) & (t \in L^-) \\ 2i\theta_j & (t \in M^-) \end{cases} \tag{2.5}$$

The solution of Riemann's boundary value problem, (2.4) and (2.5), is to be sought in a class of functions which are bounded at the ends of the segments (a_k, b_k) and which have given singularities (poles or logarithmic points). As usual, by introducing certain new functions which are obtained by removing all singularities from inside the region, the problem reduces to a Riemann boundary value problem (with other coefficients) for functions which have no singularities inside the region. The general solution of such a problem is written out in [8]. We will not introduce it here. We may note only that, in equation (1.11) of [8], it may happen that the product $[B_n(z)X_{1,2}(z)]^{-1}$ has a non-integrable singularity at certain end-points $z = g_i$ of the segments (a_k, b_k), so that equation (1.11) becomes meaningless. In that case, as is easy to see, it is necessary to take the function

$$B_n(z)\prod_i (z - g_i)^{-1}$$

instead of $B_n(z)$ in equation (1.11). The solution obtained will satisfy all conditions of the problem and the integrals will be convergent. Correspondingly, the existence conditions for the nonhomogeneous problem will be written somewhat differently. Similar remarks apply also in relation to equations (1.9) and (1.10) of [9].

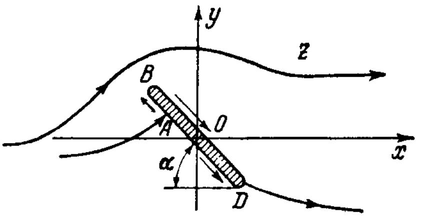

Fig. 1.

In the solution of specific problems, it is more convenient to adopt the procedure for solution given in [8,9] than to start with the general formulas.

3. Problem of the submerged wing. Consider an infinitesimally thin plate of width $2l$ which is moving with velocity v_∞ beneath a free fluid surface (Fig. 1). The fluid is considered to be ideal, incompressible, and gravity effects are absent. We take the origin of Cartesian coordinates xy at the center of the plate with the x-axis parallel to the direction of motion of the plate. The angle of inclination of the plate to the x-axis is equal to α. We reverse the motion and take the plate to be stationary and the flow approaching it with velocity v_∞. We adopt the Joukowski-Chaplygin type of flow with finite velocity at the trailing edge of the plate (Fig. 1). The problem of the submerged wing with gravity taken into account was solved for the linearized case by Lavrent'ev and Keldysh [10]. Fedorov [3] solved the problem of the submerged wing with gravity effects not taken into account, using the condition of Chaplygin and A.L. Lavrent'ev.

We map the flow region conformally in the z-plane onto the exterior of two cuts in the real axis of the ζ-plane (Fig. 2). For these, the cuts (0, 1) and (a_2, ∞) may be chosen, so that the point at infinity of the ζ-plane will correspond to the point at infinity of the z-plane [10]. The points with abscissae a and b on the upper edge of the cut (0, 1) correspond to the points A and B (Fig. 1).

We write the boundary value problem (2.2) for determining the function $\Psi(\zeta)$ in the form

$$\operatorname{Im}\frac{d\Psi}{d\zeta}=0 \quad (0,1) \quad (a_2,\infty), \qquad \Psi(\zeta)=O(\zeta^{1/2}) \quad \text{for } \zeta\to\infty \tag{3.1}$$

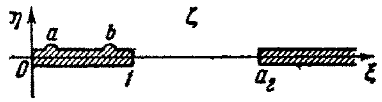

Fig. 2.

The solution of the Dirichlet problem (3.1) has the form [2,6,7]

$$\frac{d\Psi}{d\zeta}=\frac{c_0(\zeta-a)}{\sqrt{\zeta(\zeta-1)(\zeta-a_2)}} \tag{3.2}$$

Here, c_0 is a real constant; the root has its real value on the upper edge of the cut (0, 1).

The boundary value problem (2.3) for determining the function $\Phi(\zeta)$ in the case under consideration will, obviously, take the form

$$\operatorname{Re}\Phi(\zeta)=0 \qquad (a_2,\infty) \tag{3.3}$$

$\operatorname{Im}\Phi(\zeta) = -\pi+\alpha$ on the upper edge of the cut (0, 1) between the points a and b

$\operatorname{Im}\Phi(\zeta) = \alpha$ on the remaining portion of the cut (0, 1)

$\Phi(\zeta) = o(1)$ for $\zeta\to\infty$

The canonical solution of the corresponding Riemann problem is found from equation (1.10) of [8]

$$X_1(\zeta) = X_2(\zeta) = \sqrt{\zeta - a_2} \tag{3.4}$$

We take the root to be positive on the upper edge of the cut (a_2, ∞). For the case under consideration, the canonical solution is not difficult to guess. We note that here the system of equations (1.8) of [8] are satisfied, since $\alpha/\beta = 1$ (see also [9]).

The solution of the boundary value problem (3.3) will be found from formula (1.11) of [8], where the remark at the end of the last paragraph should be taken into account. Omitting fairly tedious steps connected with the calculation of the integrals, we finally obtain

$$\frac{dw}{dz} = v_\infty e^{\alpha i}\left(\frac{\sqrt{b-a_2}+\sqrt{\zeta-a_2}}{\sqrt{b-a_2}-\sqrt{\zeta-a_2}}\,\frac{\sqrt{a-a_2}-\sqrt{\zeta-a_2}}{\sqrt{a-a_2}+\sqrt{\zeta-a_2}}\right)^{1/2}\times$$
$$\times\left(\frac{2\sqrt{\zeta(\zeta-1)}\sqrt{b(b-1)}}{b-\zeta}+\frac{2\zeta(\zeta-1)}{b-\zeta}+2\zeta-1\right)^{1/2}\times$$
$$\times\left(\frac{2\sqrt{\zeta(\zeta-1)}\sqrt{a(a-1)}}{a-\zeta}+\frac{2\zeta(\zeta-1)}{a-\zeta}+2\zeta-1\right)^{-1/2} \tag{3.5}$$

Here, the function $\sqrt{[\zeta(\zeta-1)]}$ behaves like $O(\zeta)$ for $\zeta\to\infty$; the argument of the root changes by πi in going around the points a and b on the upper edge of the cut (and it is equal to $-\pi i$ on the upper edge of the cut $(0, 1)$ between the points a and b).

The mapping function $z = \omega(\zeta)$ is determined from equation

$$\frac{d\omega}{d\zeta} = \frac{d\Psi}{d\zeta} : \frac{dw}{dz} \tag{3.6}$$

The right-hand side of equation (3.6) is determined by equations (3.5) and (3.2), and is a known function of ζ.

To determine the unknown parameters c_0, a_2, a and b, we have the following conditions:

$$
\begin{aligned}
&1)\quad \arg\left(\frac{1}{v_\infty}\frac{dw}{dz}\right)\to 0 \quad \text{for } \zeta\to\infty\\
&2)\quad \oint_{C_1}\left|\frac{d\omega}{d\zeta}\right| d\zeta = 4l \qquad (C_1 \text{ is the cut } (0,\ 1))\\
&3)\quad \operatorname{Im}\int_1^{a_2}\frac{d\Psi}{d\zeta}\,d\zeta = q + v_\infty \quad (q \text{ is the parameter which determines the depth of submergence})\\
&4)\quad \oint_{C_1}\frac{d\omega}{d\zeta}\,d\zeta = 0 \qquad (\text{condition of single-valuedness})
\end{aligned}
\tag{3.7}
$$

The solution of the final system of equations (3.7) for determining the parameters c_0, a_2, a and b poses a problem in itself, due to the difficult (not tabulated) integrals which appear.

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